

The University of Chicago

CONTRIBUTIONS TO THE THEORY
OF CONJUGATE NETS

A DISSERTATION SUBMITTED TO THE FACULTY
OF THE DIVISION OF THE PHYSICAL SCIENCES
IN CANDIDACY FOR THE DEGREE OF DOCTOR
OF PHILOSOPHY

DEPARTMENT OF MATHEMATICS

1933

By
WATSON M. DAVIS

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CONTRIBUTIONS TO THE THEORY OF CONJUGATE NETS

1. Introduction. The purpose of this paper is to make some contributions to the projective differential geometry of a conjugate net on an analytic surface in ordinary space. After summarizing for subsequent use some of the theory of a conjugate net, we introduce two new invariants P and Q in connection with formulas for differentiating certain local coordinates. An application of these formulas leads to a geometric interpretation of the vanishing of P and Q .

We then define and study a certain quadric surface, a plane, a point, and two pencils of lines, all of which are covariantly associated with a point of a conjugate net of a surface. Our plane, point, and pencils of lines are analogous to the well-known canonical plane, canonical point, and canonical pencils associated with a point of a surface, but depend not only on the surface but on a given conjugate net as well. The covariant quadric plays a rôle analogous to that of the quadric of Lie in the usual surface theory.

Certain intimate relations will be found to exist between the new canonical elements and the ray-point cubic, flex-ray, and ray pencil of the given conjugate net and its associate net. A new covariant line will also be defined geometrically in a simple manner in connection with a new triple of directions given by the intersection of our quadric with the sustaining surface. Two other lines receive new geometrical characterizations. The concluding section deals with the problem of finding those reciprocal polar congruences, with respect to our quadric, for which the

developables correspond to the same curves when the axis and ray have this property.

Much of the computation needed to obtain these results will be greatly simplified by the introduction, in Section 6, of a transformation of local coordinates between the given conjugate net and its associate net.

2. Analytic Foundations. Let us consider a surface S with parametric vector equation $x = x(u, v)$. It is well known that the parametric curves on S form a conjugate net if and only if x satisfies an equation of Laplace,

$$x_{uv} = cx + ax_u + bx_v.$$

Also the first and minus-first Laplace transforms σ, ρ of the point P_x with respect to the conjugate net are defined by the formulas

$$(1) \quad \sigma = x_v - ax, \quad \rho = x_u - bx.$$

In his study of conjugate nets in ordinary space Green based his theory¹ on a system of differential equations of the form

$$(G) \quad \begin{aligned} y_{uu} &= ay_{vv} + by_u + cy_v + dy, \\ y_{uv} &= b'y_u + c'y_v + d'y. \end{aligned}$$

Recent writers² have used another system of differential equations

¹Green, "Projective Differential Geometry of One-Parameter Families of Curves, and Conjugate Nets on a Curved Surface," First Memoir, American Journal of Mathematics, Vol. 37 (1915), p. 215.

²Slotnik, "On the Projective Differential Geometry of Conjugate Nets," American Journal of Mathematics, Vol. 53 (1931), p. 143. Lane, "Conjugate Nets and the Lines of Curvature," American Journal of Mathematics, Vol. 53 (1931), p. 573.

in which the parameters enter symmetrically. In the notation of Lane this system is written in the form

$$(2) \quad \begin{aligned} x_{uu} &= px + ax_u + Ly, \\ x_{uv} &= cx + ax_u + bx_v, \\ x_{vv} &= qx + \delta x_v + Ny, \end{aligned} \quad (LN \neq 0),$$

in which the second is an equation of Laplace, and the point y is the harmonic conjugate of the point P_x with respect to the focal points of the axis of P_x , namely, the line of intersection of the two osculating planes at P_x of the two curves of the net.

The partial derivatives of y are expressed by the equations

$$(3) \quad \begin{aligned} y_u &= fx - nx_u + sx_v + Ay, \\ y_v &= gx + tx_u + nx_v + By, \end{aligned}$$

where

$$(4) \quad \begin{aligned} fN &= c_v + ac + bq - c\delta - q_u, & gL &= c_u + bc + ap - c\alpha - p_v, \\ -nN &= a_v + a^2 - a\delta - q, & tL &= a_u + ab + c - \alpha_v, \\ sN &= b_v + ab + c - \delta_u, & nL &= b_u + b^2 - b\alpha - p, \\ A &= b - (\log N)_u, & B &= a - (\log L)_v. \end{aligned}$$

System (2) has four integrability conditions³ which we shall not write here.

The Laplace-Darboux point invariants H, K , the tangential invariants $\mathcal{H}, \mathcal{K}$, the invariants $\mathcal{B}', \mathcal{C}'$ of Green, and the invariant r of Eisenhart are expressed in terms of the coefficients of equations (2) and (3) by the formulas

$$(5) \quad \begin{aligned} H &= c + ab - a_u, & K &= c + ab - b_v, \\ \mathcal{H} &= sN, & \mathcal{K} &= tL, \end{aligned}$$

³Lane, Projective Differential Geometry of Curves and Surfaces, The University of Chicago Press, p. 139.

$$(5) \quad \begin{aligned} 8\mathcal{B}' &= 4a - 2\delta + (\log r)_v, & 8\mathcal{C}' &= 4b - 2\alpha - (\log r)_u, \\ r &= N/L. \end{aligned}$$

We shall use the covariant tetrahedron with vertices at the points x , ρ , σ , y as our local tetrahedron of reference with the unit point chosen so that a point T given by an expression of the form

$$T = x_1x + x_2\rho + x_3\sigma + x_4y$$

will have local coordinates proportional to $x_1, \dots, x_4$.

Referred to this tetrahedron the quadric of Lie has the equation⁴

$$(6) \quad Lx_2^2 + Nx_3^2 + x_4(-2x_1 + 4\mathcal{C}'x_2 + 4\mathcal{B}'x_3 + kx_4) = 0$$

where

$$kN = 2(\mathcal{B}'^2 + r\mathcal{C}'^2) + \mathcal{B}'(\log \mathcal{B}'r^{1/2})_v + r\mathcal{C}'(\log \mathcal{C}'/r^{1/2})_u.$$

The equation of the ray-point cubic⁵ determined by the parametric conjugate net is

$$(7) \quad x_4 = x_1(x_2^2 + rx_3^2) - 4x_2x_3(\mathcal{B}'x_2 + r\mathcal{C}'x_3) = 0.$$

The equation of the flex-ray of this cubic is

$$(8) \quad x_4 = x_1 - \mathcal{C}'x_2 - \mathcal{B}'x_3 = 0.$$

The rays of the pencil of nets (not conjugate) given by

$$r \, dv^2 + h^2 \, du^2 = 0,$$

⁴Miss Hagen, Chicago Master's Thesis, 1926, p. 12.

⁵Lane, "Bundles and Pencils of Nets on a Surface," Transactions of the American Mathematical Society, Vol. 28 (1926), p. 149. Hagen, loc. cit., p. 15.

where h is an arbitrary constant, all pass through the point whose coordinates are proportional to

$$(9) \quad (4\mathfrak{B}'\mathfrak{C}', \mathfrak{B}', \mathfrak{C}', 0),$$

known as the center of the ray pencil.

The equation of the ray conic⁶ determined by the asymptotic net and a constant cross ratio is

$$(10) \quad x_4 = (\mathfrak{B}'^2 + r\mathfrak{C}'^2)(x_2^2 + rx_3^2) - r(x_1 - \mathfrak{C}'x_2 - \mathfrak{B}'x_3)^2 = 0.$$

3. Derivatives of the Local Coordinates. Lane has developed formulas⁷ for differentiating local coordinates referred to a certain covariant tetrahedron associated with a point on a surface referred to its asymptotic net, and has applied these formulas to the study of envelopes of certain covariant surfaces. We now seek differentiation formulas analogous to these but for coordinates referred to our local tetrahedron x, ρ, σ, γ .

It will be convenient to introduce here two invariants which have not hitherto been employed in the literature of conjugate nets. These invariants P and Q are defined by

$$(11) \quad P = f - bn + as, \quad Q = g + bt + an.$$

A geometric interpretation of the equations $P = 0$ and $Q = 0$ will be found in the next section.

In order to obtain the desired differentiation formulas let us first seek the power series expansions for the local coordinates of points in the neighborhood of the vertices of our local

⁶Lane, loc.cit., p. 163.

⁷Lane, "Canonical Configurations Associated with a Surface," Bulletin of the American Mathematical Society, Vol. 34 (1928), p. 737. Similar Formulas were given by Bomplani, "Contributi alla geometria proiettiva-differenziale di una superficie," Bullettino della Unione Matematica Italiana, Vol. 3 (1924), p. 49.

tetrahedron. We shall not need terms beyond those of the first order in the present section, but we shall compute terms of the second and third order in certain cases to be used later.

From (1), (2), (3), (5) and (11) we obtain, by differentiation and elimination, the expressions

$$\begin{aligned}
 x_u &= \rho + bx, & y_u &= Px - n\rho + s\sigma + Ay, \\
 x_v &= \sigma + ax, & y_v &= Qx + t\rho + n\sigma + By, \\
 \sigma_u &= Hx + b\sigma, & \rho_u &= -nLx + (\alpha - b)\rho + Ly, \\
 \sigma_v &= nNx + (\delta - a)\sigma + Ny, & \rho_v &= Kx + a\rho, \\
 (12) \quad x_{uu} &= (p + b\alpha)x + \alpha\rho + Ly, \\
 x_{uv} &= (c + 2ab)x + a\rho + b\sigma, \\
 x_{vv} &= (q + a\delta)x + \delta\sigma + Ny, \\
 x_{uuu} &= ()x + ()\rho + ()\sigma + (L_u + \alpha L + AL)y, \\
 x_{uuv} &= ()x + ()\rho + ()\sigma + aLy, \\
 x_{uvv} &= ()x + ()\rho + ()\sigma + bNy, \\
 x_{vvv} &= ()x + ()\rho + ()\sigma + (N_v + \delta N + BN)y,
 \end{aligned}$$

the coefficients indicated by empty parentheses being not needed for our purposes.

For points X, T, S, Y near the points x, ρ, σ, y , respectively, we find the expansions

$$\begin{aligned}
 (13) \quad X &= x + (x_u \Delta u + x_v \Delta v) + \dots, & S &= \sigma + (\sigma_u \Delta u + \sigma_v \Delta v) + \dots, \\
 T &= \rho + (\rho_u \Delta u + \rho_v \Delta v) + \dots, & Y &= y + (y_u \Delta u + y_v \Delta v) + \dots.
 \end{aligned}$$

Substituting from (12) in (13) we obtain

$$\begin{aligned}
 (14) \quad X &= x_1 x + x_2 \rho + x_3 \sigma + x_4 y, & S &= z_1 x + z_2 \rho + z_3 \sigma + z_4 y, \\
 T &= y_1 x + y_2 \rho + y_3 \sigma + y_4 y, & Y &= w_1 x + w_2 \rho + w_3 \sigma + w_4 y,
 \end{aligned}$$

where

$$x_1 = 1 + b \Delta u + a \Delta v + \dots,$$

$$x_2 = \Delta u + (\alpha \Delta u^2 + 2a \Delta u \Delta v)/2 + \dots,$$

$$x_3 = \Delta v + (2b \Delta u \Delta v + \delta \Delta v^2)/2 + \dots,$$

$$x_4 = (L \Delta u^2 + N \Delta v^2)/2 + [(L_u + \alpha L + AL) \Delta u^3 + 3aL \Delta u^2 \Delta v + 3bN \Delta u \Delta v^2 + (N_v + \delta N + BN) \Delta v^3]/6 + \dots,$$

$$(15) \quad \begin{aligned} y_1 &= -nL \Delta u + K \Delta v + \dots, & z_1 &= H \Delta u + nN \Delta v + \dots, \\ y_2 &= 1 + (\alpha - b) \Delta u + a \Delta v + \dots, & z_2 &= 0 + \dots, \\ y_3 &= 0 + \dots, & z_3 &= 1 + b \Delta u + (\delta - a) \Delta v + \dots, \\ y_4 &= L \Delta u + \dots, & z_4 &= N \Delta v + \dots, \\ w_1 &= P \Delta u + Q \Delta v + \dots, & w_3 &= s \Delta u + n \Delta v + \dots, \\ w_2 &= -n \Delta u + t \Delta v + \dots, & w_4 &= 1 + A \Delta u + B \Delta v + \dots, \end{aligned}$$

the series for y_3 and z_2 beginning with terms of the second order in Δu and Δv .

Let a point M have coordinates $x_1, \dots, x_4$ referred to the tetrahedron x, ρ, σ, y , and coordinates $\bar{x}_1, \dots, \bar{x}_4$ referred to the tetrahedron X, T, S, Y . Then the equality

$$(16) \quad x_1 x + x_2 \rho + x_3 \sigma + x_4 y = \bar{x}_1 X + \bar{x}_2 T + \bar{x}_3 S + \bar{x}_4 Y$$

is an identity in x, ρ, σ, y when the expressions (14) are substituted in the right-hand member. Equating the coefficients of x, ρ, σ, y in (16) we have the series

$$(17) \quad \begin{aligned} x_1 &= (1 + b \Delta u + a \Delta v) \bar{x}_1 - (nL \Delta u - K \Delta v) \bar{x}_2 \\ &\quad + (H \Delta u + n \Delta v) \bar{x}_3 + (P \Delta u + Q \Delta v) \bar{x}_4 + \dots, \\ x_2 &= \Delta u \bar{x}_1 + [1 + (\alpha - b) \Delta u + a \Delta v] \bar{x}_2 + (-n \Delta u + t \Delta v) \bar{x}_4 \\ &\quad + \dots, \\ x_3 &= \Delta v \bar{x}_1 + [1 + b \Delta u + (\delta - a) \Delta v] \bar{x}_3 + (s \Delta u + n \Delta v) \bar{x}_4 + \dots, \\ x_4 &= L \Delta u \bar{x}_2 + N \Delta v \bar{x}_3 + (1 + A \Delta u + B \Delta v) \bar{x}_4 + \dots. \end{aligned}$$

These equations can be solved for $\bar{x}_1, \dots, \bar{x}_4$ by interchanging $\bar{x}_1, \dots, \bar{x}_4$ and $x_1, \dots, x_4$, respectively, and changing the signs of the increments $\Delta u, \Delta v$. Holding $v = \text{constant}$ and taking the limit of $(\bar{x}_1 - x_1)/\Delta u$, ($i = 1, \dots, 4$), as Δu approaches zero, and then interchanging the roles of u and v and repeating the process we obtain the desired formulas for differentiating local point coordinates:

$$\begin{aligned}
 x_{1u} &= -bx_1 & + nLx_2 - Hx_3 - Px_4, \\
 x_{2u} &= -x_1 - (a-b)x_2 & + nx_4, \\
 x_{3u} &= & -bx_3 - sx_4, \\
 x_{4u} &= & -Lx_2 - Ax_4, \\
 (18) \quad x_{1v} &= -ax_1 - Kx_2 & - nNx_3 - Qx_4, \\
 x_{2v} &= & -ax_2 - tx_4, \\
 x_{3v} &= -x_1 & - (\delta - a)x_3 - nx_4, \\
 x_{4v} &= & -Nx_3 - Bx_4.
 \end{aligned}$$

4. Application to the Local Tetrahedron. The tangent plane to a surface S at a point P_x envelops S as P_x varies over the surface, while the planes given by $x_2 = 0$ and $x_3 = 0$ envelop the first and minus-first Laplace transform surfaces, as can be verified by application of formulas (18). We shall now seek the point at which the fourth face of our local tetrahedron, given by $x_1 = 0$, touches its envelope as P_x varies over S .

To find the characteristic line of this plane as P_x moves along a curve $v = \text{constant}$, let us differentiate its equation by means of the first of formulas (18). Thus we find that the equations of this line are

$$(19) \quad x_1 = -nLx_2 + Hx_3 + Px_4 = 0.$$

In like manner the characteristic line of this plane as P_x moves

along a curve $u = \text{constant}$ has the equations

$$(20) \quad x_1 = Kx_2 + nNx_3 + Qx_4 = 0.$$

From equations (19) the following results are evident:

The first characteristic line passes through the point P_y if and only if $P = 0$. Moreover, this line passes through the first Laplace transform P_r if and only if $H = 0$. Finally, the characteristic line passes through the minus-first Laplace transform P_p if and only if $n = 0$. Thus we have a geometric interpretation of the vanishing of the invariant P as well as new geometric interpretations of the vanishing of the invariants H and n . Similar remarks for the second characteristic line follow from equations (20).

Since this fourth face of the local tetrahedron touches its envelope, as u and v vary, at the point of intersection of these two characteristic lines, this point P_z has coordinates $z_1, \dots, z_4$ which satisfy equations (19) and (20) simultaneously, and are consequently proportional to

$$(21) \quad (0, HQ - nNP, KP + nNQ, -HK - n^2LN).$$

If $P = Q = 0$ then $z_2 = z_3 = 0$. Conversely, if $z_2 = z_3 = 0$ then $z_4 \neq 0$ and $P = Q = 0$. Hence the plane whose equation is $x_1 = 0$ touches its envelope at the point P_y if and only if $P = Q = 0$. This result could also be obtained from the expressions for y_u and y_v in (12).

This case in which the point P_z coincides with P_y is of some interest. We shall note here only two results. It can be shown that when $P = Q = 0$ the two Weingarten invariants $W^{(u)}, W^{(v)}$ are connected by the relation $W^{(u)} + W^{(v)} = 0$. Also the parametric net on the surface S_y which is the locus of P_y is a conjugate net

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if $nW(u) = 0$ (or $nW(v) = 0$). If $P = Q = n = 0$ it is easily shown that for the two parametric conjugate nets on S and S_y the first Laplace transform of one is the minus-first Laplace transform of the other.

The point P_z will be on the ray of the parametric net of the surface S if and only if

$$HK + n^2LN = 0.$$

But this relation is a necessary and sufficient condition that the focal surfaces of the ray coincide, since the focal points are given by the expression⁸

$$\rho + k\sigma$$

where k is a root of the equation

$$NHk^2 - 2nLNk - LK = 0,$$

and the discriminant of the latter is

$$4LN(n^2LN + HK).$$

Then the face of the local tetrahedron which is opposite the point P_x touches its envelope at a point on the ray of the parametric net if and only if the focal surfaces of the ray coincide.

5. Canonical Quadric of a Conjugate Net. Covariantly associated with each point of a surface are several quadric surfaces which have received considerable attention--for example, the quadric of Lie, the canonical quadric of Wilczynski, and the canonical quadric of Fubini. We shall proceed to define a quadric surface analogous to these, but covariantly associated with a point of an arbitrary conjugate net of a surface.

⁸Lane, Projective Differential Geometry of Curves and Surfaces, p. 141.

We first show that at each point P_x of a conjugate net there exists a unique quadric surface which has the following properties:

1. The quadric has second-order contact with the surface sustaining the net.

2. The axis and ray of the net at P_x are reciprocal polars with respect to the quadric.

3. The quadric passes through the point P_y which is the harmonic conjugate of P_x with respect to the focal points of the axis. The quadric that possesses these three properties will be called the canonical quadric of the conjugate net at P_x .

To find the equation of the quadrics having second-order contact with S at P_x , let us write the most general equation of a quadric surface in local coordinates and require the first four of equations (15) to satisfy this equation identically in Δu and Δv to terms of the second order. The resulting equation is

$$(22) \quad Lx_2^2 + Nx_3^2 + x_4(-2x_1 + k_2x_2 + k_3x_3 + k_4x_4) = 0,$$

where k_2, k_3, k_4 are arbitrary, being constants as long as the point P_x is fixed, and functions of u, v when P_x varies over S .

To impose the second condition it is sufficient to require that the polar planes of any two distinct points on the axis meet in the ray. But the polar plane of the point P_x is the tangent plane given by $x_4 = 0$. Let us choose P_y as the second point. The polar plane of this point with respect to the quadric given by (22) has the equation

$$-2x_1 + k_2x_2 + k_3x_3 + 2k_4x_4 = 0.$$

If this plane is to intersect the tangent plane in the ray, whose equations are $x_1 = x_4 = 0$, then $k_2 = k_3 = 0$. Hence every quadric

satisfying the first two conditions has an equation of the form

$$(23) \quad Lx_2^2 + Nx_3^2 + x_4(-2x_1 + k_4x_4) = 0.$$

Of these ∞' quadric surfaces the one which passes through P_y is characterized by the condition $k_4 = 0$. Then the equation of the canonical quadric of the parametric conjugate net is

$$(24) \quad Lx_2^2 + Nx_3^2 - 2x_1x_4 = 0.$$

It is well known that the quadric of Lie at a point P_x of the surface S touches its envelope at four points besides at P_x . These points are the vertices of the tetrahedron of Demoulin. This suggests that we should seek the points at which our canonical quadric touches its envelope as P_x varies over S .

The characteristic curve on the quadric as P_x moves along a curve $v = \text{constant}$ is found by differentiating equation (24) with respect to u , using the first four of (18) to differentiate the local coordinates. The resulting equation is

$$(L_u - 2aL + 2bL)x_2^2 + 2Px_4^2 + (A + b)(2x_1x_4 - Nx_3^2) + 2(H - \mathcal{H})x_3x_4 = 0,$$

which, with (24), gives the characteristic curve. If we eliminate x_1 between these two equations and simplify by means of equations (5) we obtain

$$(25) \quad 4\mathcal{G}'Lx_2^2 + Px_4^2 + (H - \mathcal{H})x_3x_4 = 0.$$

Evidently this equation with (24) also gives the first characteristic curve.

If $\mathcal{G}' \neq 0$ it is easily seen that the cone given by (25) is tangent to the tangent plane along the conjugate tangent for which $x_2 = x_4 = 0$. Moreover, the λ -matrix of this characteristic curve is

$$\begin{pmatrix} 0 & 0 & 0 & \lambda \\ 0 & (8\mathcal{C}' - \lambda)L & 0 & 0 \\ 0 & 0 & -\lambda N & H - \mathcal{H} \\ \lambda & 0 & H - \mathcal{H} & 2P \end{pmatrix},$$

and by the method of elementary divisors it can be shown that if $H \neq \mathcal{H}$ and $\mathcal{C}' \neq 0$ the curve is a cuspidal quartic. Since the canonical quadric and the cone have the tangent plane of S at P_x as their common tangent plane at P_x , and since this point is the vertex of the cone, it is also the cusp of the cuspidal quartic and the conjugate tangent whose equations are $x_2 = x_4 = 0$ is the cusp tangent.

In similar manner it can be shown that the characteristic curve on the canonical quadric, for P_x moving along a curve $u =$ constant, is given by (24) and

$$(26) \quad 4B'Nx_3^2 + Qx_4^2 + (K - \mathcal{K})x_2x_4 = 0.$$

If $K \neq \mathcal{K}$ and $B' \neq 0$ this second characteristic curve is also a cuspidal quartic with the other conjugate tangent, which has equations $x_3 = x_4 = 0$, as the cusp tangent.

The quadric cones given by (25) and (26) ordinarily intersect in four generators which pierce the canonical quadric at P_x and at four other points. These are the points of intersection of the two characteristic curves and hence the points at which the canonical quadric touches its envelope as P_x varies over S . We can now say that ordinarily the canonical quadric of a conjugate net touches its envelope at exactly four points besides at P_x .

Inspection of equations (25) and (26) shows that $P = 0$ and $Q = 0$ are necessary and sufficient conditions that the first

and second characteristic curves of canonical quadric, respectively, pass through the point P_y . Also, as in the case of the plane studied in Section 4, $P = Q = 0$ is a necessary and sufficient condition that the canonical quadric touch its envelope at P_y .

The following polarity will be found useful in subsequent sections of this paper. If a point with coordinates proportional to $y_1, \dots, y_4$ and a plane with coordinates proportional to $\xi_1, \dots, \xi_4$ are pole and polar with respect to the canonical quadric, then

$$(27) \quad k\xi_1 = y_4, \quad k\xi_2 = -Ly_2, \quad k\xi_3 = -Ny_3, \quad k\xi_4 = y_1,$$

where k is a proportionality factor.

6. A Transformation of Local Coordinates. We shall find it convenient in the next section to make use of the associate conjugate net,⁹ namely, the conjugate net whose tangents separate the tangents of the given conjugate net harmonically. Moreover, it will be convenient to have the transformation of local coordinates between our tetrahedron x, ρ, σ, y and the similar tetrahedron defined for the associate conjugate net. We immediately seek this transformation.

Green has found expressions¹⁰ for the coefficients of system (G) for the associate net in terms of the coefficients of the same system for the original net. In the present notation these formulas become

$$(28) \quad \begin{aligned} \bar{r} &= \phi_v^2 / \psi_v^2, & \psi_v^2 \bar{q} - \phi_v^2 \bar{p} &= r^{1/2} c, & 4\phi_v \psi_v \bar{c} &= q - rp, \\ \phi_v^2 \bar{a} &= r^{1/2} \phi_{uv} + (a - r^{1/2} b) \phi_v, & \psi_v^2 \bar{d} &= -r^{1/2} \psi_{uv} + \\ & & & & (a + r^{1/2} b) \psi_v. \end{aligned}$$

⁹Green, Second Memoir, American Journal of Mathematics, Vol. 38 (1916), p. 313.

¹⁰Ibid., p. 316.

$$(28) \quad 2\phi_v \bar{b} = a - 2\mathfrak{B}' - r^{1/2}(b - 2\mathfrak{C}'), \quad 2\psi_v \bar{a} = a - 2\mathfrak{B}' + r^{1/2}(b - 2\mathfrak{C}'),$$

where the transformation

$$(29) \quad \bar{u} = \phi(u, v), \quad \bar{v} = \psi(u, v),$$

makes the associate net parametric in parameters $\bar{u}$, $\bar{v}$, and where

$$(30) \quad r^{1/2}\phi_u + \phi_v = 0, \quad r^{1/2}\psi_u - \psi_v = 0.$$

Useful relations among the partial derivatives of ϕ and ψ are

$$(31) \quad \begin{aligned} \phi_{vv} &= -r^{1/2}\phi_{uv} + (\log r)_v \phi_v/2, \\ r\phi_{uu} &= -r^{1/2}\phi_{uv} + r^{1/2}(\log r)_u \phi_v/2, \\ \psi_{vv} &= r^{1/2}\psi_{uv} + (\log r)_v \psi_v/2, \\ r\psi_{uu} &= r^{1/2}\psi_{uv} - r^{1/2}(\log r)_u \psi_v/2. \end{aligned}$$

Also the application of (28) in (5) yields the simple relations

$$(32) \quad 2\psi_v \bar{\mathfrak{B}}' = -r^{1/2}\mathfrak{C}' - \mathfrak{B}', \quad 2\phi_v \bar{\mathfrak{C}}' = r^{1/2}\mathfrak{C}' - \mathfrak{B}'.$$

In addition to the relations (28) we shall need $\phi_v^{2\bar{p}} + \psi_v^{2\bar{q}}$ expressed in terms of the coefficients of system (2). By using equations (3) and (28) we can find, after some reduction, the relation

$$\begin{aligned} 4(\phi_v^{2\bar{p}} + \psi_v^{2\bar{q}}) &= 2a_v + 2rb_u - 2a^2 - 2rb^2 - 4\mathfrak{B}'_v - 4r\mathfrak{C}'_u \\ &\quad + r(b - 2\mathfrak{C}')(\log r)_u - (a - 2\mathfrak{B}')(\log r)_v \\ &\quad + 8\mathfrak{B}'^2 + 8\mathfrak{C}'^2. \end{aligned}$$

In order to obtain the transformation of local coordinates let us consider a point M whose local coordinates are $x_1, \dots, x_4$ referred to the local tetrahedron x, ρ, σ, γ of the original net and $\bar{x}_1, \dots, \bar{x}_4$ referred to the local tetrahedron $x, \bar{\rho}, \bar{\sigma}, \bar{\gamma}$ of

the associate net. Then when ρ , σ , y are expressed in terms of x , $\bar{\rho}$, $\bar{\sigma}$, $\bar{y}$ the relation

$$(34) \quad \bar{x}_1 x + \bar{x}_2 \bar{\rho} + \bar{x}_3 \bar{\sigma} + \bar{x}_4 \bar{y} = x_1 x + x_2 \rho + x_3 \sigma + x_4 y$$

is an identity in x , $\bar{\rho}$, $\bar{\sigma}$, $\bar{y}$.

In view of (29) we have, by differentiation, the formulas

$$(35) \quad \begin{aligned} x_u &= x_u \phi_u + x_{\bar{v}} \psi_u, & x_v &= x_u \phi_v + x_{\bar{v}} \psi_v, \\ x_{vv} &= x_{uu} \phi_v^2 + 2x_{u\bar{v}} \phi_v \psi_v + x_{\bar{v}\bar{v}} \psi_v^2 + x_u \phi_{vv} + x_{\bar{v}} \psi_{vv}, \end{aligned}$$

and from the last of (2) we have

$$(36) \quad Ny = x_{vv} - qx - \delta x_v.$$

If we substitute from (35) in (1) and (36) and make certain reductions using system (2) written for the associate net, and equations (28), (31) and (33), we obtain the relations

$$(37) \quad \begin{aligned} r^{1/2} \rho &= -2r^{1/2} \mathfrak{C}' x - \phi_v \bar{\rho} + \psi_v \bar{\sigma}, \\ \sigma &= -2\mathfrak{B}' x + \phi_v \bar{\rho} + \psi_v \bar{\sigma}, \\ Ny &= [\mathfrak{B}'(\log r)_v/2 - r\mathfrak{C}'(\log r)_u/2 - \mathfrak{B}'_v - r\mathfrak{C}'_u \\ &\quad - 2(\mathfrak{B}'^2 + r\mathfrak{C}'^2)]x + 2\phi_v(\mathfrak{B}' - r^{1/2}\mathfrak{C}')\bar{\rho} \\ &\quad + 2\psi_v(\mathfrak{B}' + r^{1/2}\mathfrak{C}')\bar{\sigma} + 2\bar{N}\psi_v^2\bar{y}. \end{aligned}$$

Substituting from (37) in (34) and equating coefficients of x , $\bar{\rho}$, $\bar{\sigma}$, $\bar{y}$ we obtain the desired transformation of local coordinates, namely,

$$(38) \quad \begin{aligned} \bar{x}_1 &= x_1 - 2\mathfrak{C}'x_2 - 2\mathfrak{B}'x_3 + [\mathfrak{B}'(\log r)_v - r\mathfrak{C}'(\log r)_u \\ &\quad - 2(\mathfrak{B}'_v + r\mathfrak{C}'_u) - 4(\mathfrak{B}'^2 + r\mathfrak{C}'^2)]x_4/2N, \\ \bar{x}_2 &= [-x_2/r^{1/2} + x_3 + (\mathfrak{B}' - r^{1/2}\mathfrak{C}')2x_4/N]\phi_v, \\ \bar{x}_3 &= [x_2/r^{1/2} + x_3 + (\mathfrak{B}' + r^{1/2}\mathfrak{C}')2x_4/N]\psi_v, \\ \bar{x}_4 &= 2\psi_v^2\bar{N}x_4/N. \end{aligned}$$

As an application of the transformation just derived we can find the equation of the canonical quadric of the associate net, which we shall call the associate canonical quadric. Its equation referred to the tetrahedron $x, \bar{p}, \bar{\sigma}, \bar{y}$ is of course

$$\overline{Lx}_2^2 + \overline{Nx}_3^2 - 2\overline{x}_1\overline{x}_4 = 0.$$

Substituting from (38) and (28) in this equation we obtain

$$(39) \quad Lx_2^2 + Nx_3^2 + x(-2x_1 + 8\mathcal{C}'x_2 + 8\mathcal{B}'x_3 + kx_4) = 0,$$

where

$$Nk = r\mathcal{C}'(\log r)_u - \mathcal{B}'(\log r)_v + 2(r\mathcal{C}'_u + \mathcal{B}'_v) + 8(r\mathcal{C}'^2 + \mathcal{B}'^2).$$

7. Canonical Plane and Point of a Conjugate Net. Covariantly associated with each point P_x of a surface S are a plane and a point known as the canonical plane and canonical point, respectively. The projective normal of Fubini, the directrices of Wilczynski, the edges of Green, and the axes of Čech are members of the pencils of lines in the canonical plane with center at P_x , and in the tangent plane at P_x with center at the canonical point. We now define a plane and a point analogous to these but covariantly associated with each point P_x of a conjugate net on S .

We shall call the point of intersection of the ray and the associate ray the canonical point of the conjugate net. The associate ray is given by $\bar{x}_1 = \bar{x}_4 = 0$, or, on applying transformation (38), by

$$(40) \quad x_4 = x_1 - 2(\mathcal{C}'x_2 + \mathcal{B}'x_3) = 0.$$

If we solve these equations with $x_1 = 0$ we find the coordinates¹¹

¹¹Ibid., p. 318.

of the canonical point of the parametric conjugate net (and of its associate net) proportional to

$$(41) \quad (0, \mathfrak{B}', -\mathfrak{C}', 0).$$

The plane determined by the axis and the associate axis will be called the canonical plane of the conjugate net. The associate axis is given by $\bar{x}_2 = \bar{x}_3 = 0$, or, if we use our transformation of local coordinates, by

$$(42) \quad Lx_2 + 2\mathfrak{C}'x_4 = Nx_3 + 2\mathfrak{B}'x_4 = 0.$$

The equation of the plane through this line and the axis of the original net is

$$(43) \quad \mathfrak{B}'x_2 - r\mathfrak{C}'x_3 = 0.$$

This is the equation of the canonical plane of the parametric net and its associate net.

It is easily verified that the coordinates (41) of the canonical point and the coordinates $(0, \mathfrak{B}', -r\mathfrak{C}', 0)$ of the canonical plane satisfy the equations (27). Then the canonical point and the canonical plane are pole and polar with respect to the canonical quadric. By symmetry they are also pole and polar with respect to the associate canonical quadric.

The pencil of lines in the canonical plane with center at P_x will be called the first canonical pencil of the net, and the pencil of lines in the tangent plane at P_x with center at the canonical point will be called the second canonical pencil of the net. The flex-ray, given by equations (8), belongs to the second canonical pencil.

We shall call the line of intersection of the canonical plane with the tangent plane the first canonical tangent, and the

line joining the canonical point to the point P_x the second canonical tangent. For our parametric conjugate net and its associate net the equations of the first and second canonical tangents are, respectively,

$$(44) \quad x_4 = \mathfrak{B}'x_2 - r\mathfrak{C}'x_3 = 0,$$

and

$$(45) \quad x_4 = \mathfrak{C}'x_2 + \mathfrak{B}'x_3 = 0.$$

We shall now show that the canonical plane and canonical point are closely related to the ray-point cubic and center of the ray pencil. If in equations (7) we set $x_1 = 0$, we find that the ray-point cubic intersects the ray at ρ and σ , and also at the point given by

$$(46) \quad (0, r\mathfrak{C}', -\mathfrak{B}', 0).$$

The first canonical tangent intersects the ray in the point whose coordinates are proportional to

$$(47) \quad (0, r\mathfrak{C}', \mathfrak{B}', 0).$$

Evidently the two points given by (46) and (47) separate the Laplace transforms, ρ and σ , harmonically.

Moreover, it can also be shown that the point with coordinates (46) is the center of the associate ray pencil. For, if we substitute from (46) in (38) and then apply relations (32), we can obtain the coordinates

$$(4\bar{\mathfrak{B}}'\bar{\mathfrak{C}}', \bar{\mathfrak{B}}', \bar{\mathfrak{C}}', 0).$$

But these coordinates are precisely of the form of (9) written for the associate net. We can state these results as follows:

The point in which the ray-point cubic intersects the ray (besides ρ and σ) is the center of the associate ray pencil, and this point is the harmonic conjugate of the point of intersection of the first canonical tangent and the ray with respect to the Laplace transforms.

8. Reciprocal Congruences and Canonical Pencils. In the theory of a surface in ordinary space referred to its asymptotic net the notion of congruences of lines which are reciprocal polars with respect to any quadric of Darboux plays an important rôle. We shall now consider congruences of pairs of lines which are reciprocal polars with respect to the canonical quadric of a conjugate net of a surface, when one of the lines χ_1 passes through P_x but does not lie in the tangent plane, and the other line χ_2 lies in the tangent plane but does not pass through P_x . In this section we shall confine our attention to such reciprocal polar lines as belong to the canonical pencils.

Analytically the line χ_1 which goes through P_x and does not lie in the tangent plane can be determined by assigning a point on χ_1 whose coordinates are proportional to

$$(48) \quad (0, -d, -e, 1).$$

By the relations (27) the line χ_2 which is the reciprocal polar of χ_1 with respect to the canonical quadric is found to have the equations

$$(49) \quad x_4 = x_1 + dLx_2 + eNx_3 = 0.$$

The points ξ and η in which χ_2 crosses the conjugate tangents are given by the equations

$$(50) \quad \xi = \rho - dLx, \quad \eta = \sigma - eNx.$$

The lines λ_1 and λ_2 will be lines of the first and second canonical pencils, respectively, if

$$(51) \quad dL = h\mathfrak{C}', \quad eN = h\mathfrak{B}',$$

where h is a parameter.

For $h = 0$ the line λ_1 is the axis of the parametric conjugate net and λ_2 is the ray of the same net; for $h = 2$ the line λ_1 is the associate axis; for $h = -1$ the line λ_2 is the flex-ray; for $h = -2$ the line λ_2 is the associate ray; and for $h = \infty$ the lines λ_1 and λ_2 are the first and second canonical tangents, respectively. We note that any pair of lines, λ_1 and λ_2 , given by the same value of h in (51) are reciprocal polars with respect to the canonical quadric. Moreover, it is seen that the associate axis and associate ray are not corresponding lines, since they are given by $h = 2$ and $h = -2$, respectively.

The flex-ray, which is the line of the three collinear inflections of the ray-point cubic, can be characterized in another manner. In fact, if the sustaining surface is not ruled the only line in the tangent plane not through P_x which intersects each conjugate tangent in the pole of the other conjugate tangent with respect to the ray conic is the flex-ray. For the point ξ given by (50) has the line with equations

$$x_4 = (\mathfrak{B}'^2 + r\mathfrak{C}'^2)x_2 + r(dL + \mathfrak{C}')(x_1 - \mathfrak{C}'x_2 - \mathfrak{B}'x_3) = 0$$

as its polar with respect to the ray conic given by (10). If $\mathfrak{B}'^2 + r\mathfrak{C}'^2 = 0$ the surface S is ruled.¹² If $\mathfrak{B}'^2 + r\mathfrak{C}'^2 \neq 0$ this polar line will be the conjugate tangent given by $x_2 = x_4 = 0$ if and only if $dL + \mathfrak{C}' = 0$. Likewise if $\mathfrak{B}'^2 + r\mathfrak{C}'^2 \neq 0$, the polar of the point η with respect to the ray conic will be the other

¹²Green, First Memoir, p. 239.

conjugate tangent in case $eN + \mathfrak{B} = 0$. But for d and e determined by these relations equations (49) become those of the flex-ray.

The second canonical tangent was defined simply as the line joining the canonical point and the point P_x . We can also characterize this line in a new way. Of the ∞^3 quadric surfaces (22) having second-order contact consider two distinct quadric surfaces, where k_2, k_3, k_4 have values k_2', k_3', k_4' and k_2'', k_3'', k_4'' in the two cases, and where not both of the relations $k_2' = k_2'', k_3' = k_3''$ hold. Every such pair of quadrics intersect in the asymptotic tangents and a residual conic which is not in the tangent plane. Consider in particular the plane of such a conic defined by the quadric of Lie and the canonical quadric of the parametric conjugate net. Its equation is obtained by subtracting (24) from (6) and disregarding the factor x_4 . We thus obtain the equation

$$(52) \quad 8N(\mathfrak{C}'x_2 + \mathfrak{B}'x_3) + [4(\mathfrak{B}'^2 + r\mathfrak{C}'^2) + 2(\mathfrak{B}'_v + r\mathfrak{C}'_u) + \mathfrak{B}'(\log r)_v - r\mathfrak{C}'(\log r)_u]x_4 = 0.$$

In similar manner the quadric of Lie and the associate canonical quadric determine the plane whose equation is

$$(53) \quad 8N(\mathfrak{C}'x_2 + \mathfrak{B}'x_3) + [12(\mathfrak{B}'^2 + r\mathfrak{C}'^2) + 2(\mathfrak{B}'_v + r\mathfrak{C}'_u) - 3\mathfrak{B}'(\log r)_v + 3r\mathfrak{C}'(\log r)_u]x_4 = 0,$$

and the canonical quadric and associate canonical quadric determine the plane given by

$$(54) \quad 8N(\mathfrak{C}'x_2 + \mathfrak{B}'x_3) + [8(\mathfrak{B}'^2 + r\mathfrak{C}'^2) + 2(\mathfrak{B}'_v + r\mathfrak{C}'_u) - \mathfrak{B}'(\log r)_v + r\mathfrak{C}'(\log r)_u]x_4 = 0.$$

To find the intersections of each of these planes with the tangent plane we set $x_4 = 0$ and have, in each case, the equa-

tions (45) of the second canonical tangent. We may now state that the three planes of the residual conics of intersection of the quadric of Lie, the canonical quadric, and the associate canonical quadric, taken in pairs, all intersect the tangent plane in the second canonical tangent.

Moreover, the last one of these three planes and the tangent plane separate the other two harmonically. For, if we designate the left member of (54) by F and give the pencil of planes by $F + px_4 = 0$, then (54) and the tangent plane are given by $p = 0$ and $p = \infty$, respectively, while the parameter values for the other two planes are $p = \pm t$, where

$$t = 2\mathfrak{B}'(\log r)_v - 2r\mathfrak{C}'(\log r)_u - 4(\mathfrak{B}' + r\mathfrak{C}').$$

At each point of a surface S are three tangents known as the tangents of Darboux and three related tangents known as the tangents of Segre, which have appeared many times in the literature. We now define three analogous tangents at each point of a conjugate net of S . All the quadric surfaces given by (23) have second-order contact with S at P_x and intersect S in a curve with a triple point at P_x . The triple point tangents of this curve will be called the triple tangents of the conjugate net at the point P_x .

To find the directions of the triple tangents for our parametric conjugate net we can substitute the first four of series (15) in (23) and set the term of third order in Δu and Δv equal to zero. After some reduction we get the equation

$$\mathfrak{C}'du^3 + r\mathfrak{B}'dv^3 = 0.$$

The equations of the triple tangents are

$$x_4 = \mathfrak{C}'x_2^3 + r\mathfrak{B}'x_3^3 = 0.$$

We next show that the triple tangents intersect the ray-point cubic in three collinear points, and that the line of these three points is a line of the second canonical pencil. Any one of the three triple tangents is given by the equations

$$(56) \quad x_4 = \mathcal{C}'^{1/3}x_2 + \epsilon r^{1/3}\mathcal{B}'^{1/3}x_3 = 0, \quad (\epsilon^3 = 1).$$

If we solve these equations with (7) we find that the coordinates of the points where the triple tangents intersect the ray-point cubic are proportional to

$$(57) \quad \begin{aligned} & [\mathcal{B}'^{1/3}\mathcal{C}'^{1/3}(\mathcal{B}'^{2/3} - r^{1/3}\mathcal{C}'^{2/3}), -r^{1/3}\mathcal{B}'^{1/3}, \mathcal{C}'^{1/3}], \\ & [4\mathcal{B}'^{1/3}\mathcal{C}'^{1/3}(\mathcal{B}'^{2/3} - \omega r^{1/3}\mathcal{C}'^{2/3}), -\omega r^{1/3}\mathcal{B}'^{1/3}, \mathcal{C}'^{1/3}], \\ & [4\mathcal{B}'^{1/3}\mathcal{C}'^{1/3}(\mathcal{B}'^{2/3} - \omega^2 r^{1/3}\mathcal{C}'^{2/3}), -\omega^2 r^{1/3}\mathcal{B}'^{1/3}, \mathcal{C}'^{1/3}], \end{aligned}$$

where ω is a complex cube root of unity and it is understood that the coordinate x_4 is always zero. The determinant of these coordinates is identically zero, so the three points are collinear.

The equations of the straight line through these three points are found to be

$$(58) \quad x_4 = x_1 - 4(\mathcal{C}'x_2 + \mathcal{B}'x_3) = 0.$$

This is a line of the second canonical pencil for which $h = -4$. We may refer to this line as the line of collineation of the net.

Let us consider any line of the second canonical pencil which is covariantly associated with the conjugate net, and given by $h = k$. There is a corresponding line covariantly associated with the associate net in the same way, whose equations are

$$\bar{x}_4 = \bar{x}_1 + k(\bar{\mathcal{C}}'\bar{x}_2 + \bar{\mathcal{B}}'\bar{x}_3) = 0,$$

referred to the local tetrahedron of the associate net. Under transformation (38) this becomes

$$(59) \quad x_4 = x_1 - (k + 2)(\mathcal{C}'x_2 + \mathcal{B}'x_3) = 0.$$

Applying this result in the case $k = -4$ we find that the associate line of collineation is the line of the second canonical pencil for which $h = 2$. But for this value of h we have seen that the corresponding line of the first canonical pencil is the associate axis. Therefore the associate line of collineation is the reciprocal polar of the associate axis with respect to the canonical quadric.

We note that all the following six lines are members of the second canonical pencil: The ray, the associate ray, the flex-ray, the second canonical tangent, the line of collineation, and the associate line of collineation. Besides these lines Lane has defined and studied another line called the principal join,¹³ which also belongs to the second canonical pencil. In our notation it is given by $h = -8/3$. The associate principal join is given by $h = 2/3$.

9. Reciprocal Congruences with Coincident Curves. It is known that if $n \neq 0$ the ray and axis curves coincide if and only if $H = \mathcal{H}$ and $K = \mathcal{K}$. In this concluding section we shall seek other reciprocal polar lines λ_1 and λ_2 which have coincident curves when $H = \mathcal{H}$ and $K = \mathcal{K}$.

Any point on λ_1 , besides P_x , is given by

$$(60) \quad z = w + \lambda x, \quad (\lambda \text{ scalar}),$$

the coordinates of the point w being given by (48). A curve on the surface S , given by the equations

$$u = u(t), \quad v = v(t),$$

¹³Lane, "Invariants of Intersection of Two Curves on a Surface," American Journal of Mathematics, Vol. 54 (1932).

along which the lines χ_1 will be arranged in developable surfaces can be determined by requiring z' to be on χ_1 , where the prime denotes differentiation with respect to t . After some computation we obtain the relation

$$(61) \quad \begin{aligned} z' = & \left[()x + (R_1 + \lambda)\rho + S_1\sigma + (A - dL)w \right] u' \\ & + \left[()x + R_2\rho + (S_2 + \lambda)\sigma + (B - eN)w \right] v', \end{aligned}$$

where

$$(62) \quad \begin{aligned} R_1 &= d(b - a + A - dL) - d_u - n, \\ S_1 &= -e(b - A + dL) - e_u + s, \\ R_2 &= -d(a - B + eN) - d_v + t, \\ S_2 &= e(a - \delta + B - eN) - e_v + n. \end{aligned}$$

If the point z' is on χ_1 the coefficients of ρ and σ in (61) must vanish. We then have the conditions

$$(63) \quad (R_1 + \lambda)du + R_2 dv = 0, \quad S_1 du + (S_2 + \lambda)dv = 0.$$

Eliminating λ we obtain

$$(64) \quad S_1 du^2 + (S_2 - R_1)du dv - R_2 dv^2 = 0,$$

which is the equation of the curves along which the congruence of lines χ_1 are arranged in developable surfaces.

Let us also find the corresponding curves for the congruence of lines χ_2 . Using the same method we can require the point ζ' to be on χ_2 , where $\zeta = \xi + \mu\eta$, (μ scalar), and where ξ and η are given by equations (50). It is found that the congruence of lines χ_2 are arranged in developable surfaces along curves given by the equation

$$(65) \quad LU_1 du^2 + (LU_2 - NT_1)du dv - NT_2 dv^2 = 0,$$

U_1, U_2, T_1, T_2 being defined by the relations

$$\begin{aligned}
 (66) \quad U_1 &= H - (eN)_u - deLN, \\
 U_2 &= nN - (eN)_v + e\delta N - e^2N^2 - 2aeN, \\
 T_1 &= -nL - (dL)_u + d\alpha L - d^2L^2 - 2bdL, \\
 T_2 &= K - (dL)_v - deLN.
 \end{aligned}$$

We may now ask for the conditions under which the curves given by (64) and those given by (65) coincide when $H = \mathcal{H}$ and $K = \mathcal{K}$. If we multiply equation (64) by LN it is easily seen that the coefficients of du^2 and the coefficients of dv^2 in the two equations are identically equal if $H = \mathcal{H}$ and $K = \mathcal{K}$. For the curves to coincide it is then also necessary that the coefficients of $du dv$ be equal, i.e., that

$$LJ_2 - NT_1 = (S_2 - R_1)LN.$$

This condition reduces to

$$e\mathcal{B}' = d\mathcal{C}'$$

or

$$(67) \quad d = k\mathcal{B}', \quad e = k\mathcal{C}',$$

where k is a proportionality factor.

It is evident that lines $\mathcal{L}_1$ for which (67) hold form a pencil of lines in the plane with the equation

$$\mathcal{C}'x_2 - \mathcal{B}'x_3 = 0.$$

But this plane can be characterized in the following manner: It is the plane determined by the axis and the harmonic conjugate of the canonical point with respect to the Laplace transforms. Let us call the pencil of lines in this plane with center at the point P_x the first anti-canonical pencil.

The lines λ_2 for which (67) hold form a pencil of lines in the tangent plane with its center at the point given by (46). But we have seen that this is the center of the pencil of lines which we have called the associate ray pencil.

We can now state the answer to our question in the following way: If $H = \mathcal{H}$ and $K = \mathcal{K}$ lines λ_1 and λ_2 given by (48) and (49), which are reciprocal polars with respect to the canonical quadric, generate congruences whose developables correspond to the same curves if and only if the line λ_2 belongs to the associate ray pencil. Then also the line λ_1 belongs to the first anti-canonical pencil.

VITA

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